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On Vlasov–Manev Equations. I: Foundations, Properties, and Nonglobal Existence

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We consider the classical stellar dynamic (Vlasov) equation with a so-called Manev correction (based on a pair potential $\gamma/r + \varepsilon/r^2$). For the pure Manev potential $\gamma = 0$ we discuss both the continuous case and the N-body problem and show that global solutions will not exist if the initial energy is negative. Certain global solutions can be constructed from local ones by a transformation which is peculiar for the ε/r^2 law. Moreover, scaling arguments are used to show that Boltzmann collision terms are meaningful in conjunction with Manev force terms. In an appendix, a formal justification of the Manev correction based on the quasirelativistic Lagrangian formalism for the motion of a particle in a central force field is given.

KEY WORDS: Vlasov–Poisson equations; Manev correction.

1. INTRODUCTION AND BACKGROUND

One of the first and best-known confirmations of Einstein's theory of general relativity was the qualitatively correct explanation of the advance of the perihelion of Mercury by 43 arc seconds per century, a fact which was unexplained by Newton's laws and classical mechanics.

There is, however, an alternative (and related) way to produce this effect. In a series of papers published between 1924 and 1930, G. Manev [Ma1–Ma4] studied a correction to the attractive Newtonian potential of the type

$$U(r) = -\frac{\gamma}{r} - \frac{\varepsilon}{r^2} \tag{1.1}$$

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where $\gamma = G(m_1 + m_2)$, (G is the gravitational constant, m_1, m_2 are the masses of two mutually attracting bodies, and r is the distance between them) and $\gamma = 3\gamma^2/c^2$ (see Diacu *et al.* [D] for a detailed introduction to and discussion of the potential given in (1.1)). We shall henceforth refer to (1.1) as a "Manev" potential, with $U_n(r) := -\gamma/r$ as the "Newtonian" part and $U_m(r) := -\varepsilon/r^2$ the "Manev" part.

Indeed, as is well known, solutions of the two-body problem with an attractive force given by a potential as in (1.1) lead to precessional ellipses rather than to stationary ellipses, and the above choice of the constant ε gives the correct rate of precession for Mercury. For a detailed discussion of the history and applications of the Manev law, we refer the reader to [D]. A most remarkable fact is that already Newton was aware of precessional ellipses as solutions to the simplified two-body problem (see [N]); the full two-body problem is solved in [D].

We give a rationale for a correction of Manev type to the Newtonian potential in the appendix, but we emphasize that we only consider this as a possible mathematical model mimicking some relativistic effects, without any hard qualitative or quantitative evidence.

The purpose of the present article is to consider the Manev gravitational law in the context of stellar dynamics. The classical stellar dynamical equation, first introduced by Jeans in 1915, treats the evolution of a stellar system as given by

$$\partial_t f + v \cdot \nabla_x f + E_1[\rho] \cdot \nabla_v f = 0 \tag{1.2}$$

where $f(x, v, t)$ is the stellar density at time t, position x and velocity v, $\rho(x, t) = \int f(x, v, t)\, dv$ is the spatial density, and

$$U_n[\rho](x, t) = -\gamma \int \frac{\rho(y, t)}{|x - y|}\, dy \tag{1.3}$$

$$E_1[\rho](x, t) = -\nabla_x U_n[\rho](x, t) \tag{1.4}$$

The potential U_n arises by linear superposition of the Newtonian potentials of all the stellar bodies defining the density ρ.

The system (1.2–4)is a classical and fairly simple model for the evolution of globular clusters, galaxies and galactic dust clouds (all stars are assumed to be equal, which they are clearly not; effects like double stars etc., are ignored). A famous result first obtained by Pfaffelmoser [Pf] states that the initial value problem for the equations (1.2–4) has a global classical solution; specifically, if f_0 is C^1 and has compact support, then (1.2–4) admits a unique solution such that $f(\cdot, t)$ is C^1 and has compact support for all $t \geqslant 0$ (for a more transparent version of the proof see Schaeffer [Sch]; for a different proof see Lions and Perthame [LP]. A good recent reference for these results is Glassey [G]).

The global existence results from [Pf] and [LP] were a major breakthrough in the mathematical theory of plasma and stellar dynamics (for the plasma physical case, all one has to do is consider $\gamma < 0$ in (1.3), i.e., switch to repulsive forces; the existence results from [Pf] and [LP] cover both cases). However, these results possess one unsatisfying aspect. In real stellar evolution, many astrophysicists believe in gravitational collapse; even though we can't see them, we expect black holes at the centres of large galaxies and elsewhere, but the solutions of the system (1.2–4) predict no such gravitational singularities.

As we shall see in Section 2, the stellar dynamical equation with a Manev-type gravitational correction of the type (1.1) must be expected to lead to singularities; for general initial data, no global solution will exist.

The potential law (1.1) has an interesting property, which we believe to be related to the non-global existence just mentioned. Consider the N-body problem for the Newtonian (N) and Manev (M—see (1.1)) attractive potentials respectively. It is a classical conjecture that the $6N$-dimensional Lebesgue measure of phase points whose evolution under (N) leads to collision singularities is zero (this is easily seen for $N = 2$, see the appendix); in contrast, it is known (see [Sa] or [D]) that the measure of the set of such phase points for (M) is positive. In [D], this property is derived by a careful analysis of the explicit solution of the two-body problem. As a side result, we present in Section 2 an indirect argument which shows that, for general but fixed N, the gravitational law (1.1) will lead to singularities from initial sets with positive measure.

This article is organized as follows. In Section 2, we introduce Manev equations and discuss the existence of the correction term relative to the classical equation. We present the invariants and use them to argue that global existence is not to be expected in the attractive case; the argument we employ was first suggested by E. Horst [H] for the classical Vlasov equation in 4 or more space dimensions. An adaptation of this argument also proves this property for the N-body problem of (1.1), which we mentioned in the previous paragraph. In Section 3, we present a rescaling property for the pure Manev equation (i.e., the case where $\gamma = 0$). This rescaling property can be used to transform local solutions to global ones; in this process, the initial values transform in such a way that the energy changes from negative to positive.

In Section 4, we discuss the well-posedness of the initial value problem from a formal point of view. We demonstrate that zero temperature data lead to an ill-posed initial value problem, and we use classical linear stability analysis for the Euler equations associated with generalized Vlasov equations, and for the full Vlasov equation with the pure Manev potential.

In Section 5, we investigate whether the effect of Manev-type potentials in a particle system best be modeled by the Vlasov-Manev force term, or by a Boltzmann collision term. We find the interesting result that potentials of type α/r^n lead to Vlasov-type collision terms for $n<2$, to Boltzmann collision terms for $n>2$, and to both together exactly if $n=2$. We mention that Vlasov-Boltzmann equations have been discussed by Lions [L]. Our discussion suggests that for Newtonian force terms, the Boltzmann collision term should be formally negligible relative to the Vlasov term.

Part II of our work will address the local existence and uniqueness question for Vlasov–Manev systems. In view of the results in Section 4, it is not surprising that such existence results are not easy to prove. Specifically, we need a collection of estimates of the singular integrals defining the Vlasov–Manev force terms. For the convenience of the reader, we list these estimates in an appendix, with sketches of proofs.

Part III of our work will deal in some more detail with the derivation of kinetic equations for particles which interact via an attractive potential of Manev type as given by (1.1). We find that such equations should include Vlasov-Manev force terms, Boltzmann collision operators and a Smoluchowski type coagulation integral.

2. VLASOV–MANEV EQUATIONS

As already described in Section 1, the stellar dynamic Manev equation (SM) arises from the standard stellar dynamic equation by adding a correction to the force field as in (1.1). The result is an equation

$$\partial_t f + v \cdot \nabla_x f + (E_1[\rho] + E_2[\rho]) \cdot \nabla_v f = 0 \tag{2.1}$$

with

$$E_1[\rho](x,t) = -\nabla_x U_n[\rho](x,t) = -\gamma \int \frac{x-y}{|x-y|^3}\, \rho(y,t)\, dy \tag{2.2}$$

$$\left(U_n[\rho](x,t) = -\gamma \int \frac{\rho(y,t)}{|x-y|}\, dy \right)$$

and

$$E_2[\rho](x,t) = -\nabla_x U_m[\rho](x,t) = -2\varepsilon \int \frac{x-y}{|x-y|^4}\, \rho(y,t)\, dy \tag{2.3}$$

In (2.3), the integral only exists as a Cauchy principal value; indeed, the transformation defined by $E_2[\rho]$ is a multiple of the Riesz transform, a well-known singular integral operator (see, e.g., [St], pp. 117–121). It is a bounded linear operator in L^p for $1 < p < \infty$, but not in L^1 or L^∞. For our purposes, we will use the fact that $E_2[\rho]$ is well defined if ρ is Hölder continuous with exponent $0 < \alpha < 1$ and in L^1.

In that case,

$$E_2[\rho](x) = -2\varepsilon \left\{ \lim_{\delta \searrow 0} \int_{\delta < |y-x| < R} \frac{x-y}{|x-y|^4} (\rho(y) - \rho(x))\, dy + \int_{|y-x| \geqslant R} \frac{x-y}{|y-x|^4} \rho(y)\, dy \right\}$$

and it is easy to see that this definition is independent of the choice of $R > 0$. The first term on the right exists in view of the Hölder continuity of ρ. We shall mainly be concerned with the cases $\gamma > 0$, $\varepsilon > 0$, or $\gamma = 0$, $\varepsilon > 0$ (we refer to these cases as stellar dynamic Manev equation (SM) or pure stellar dynamic Manev equation (PSM)). We focus on these cases because of the particularly interesting mathematics, but mention that the cases $\gamma \geqslant 0$, $\varepsilon < 0$;$\gamma \leqslant 0$, $\varepsilon < 0$; and $\gamma \leqslant 0$, $\varepsilon > 0$ can all be considered. Our local existence theory from part II applies to all cases.

2.1. Invariants

In the remainder of this section, we assume that $f(x, v, t)$ is a smooth classical solution of (2.1).

First, recall the solution strategy for the ordinary Vlasov equation, which applies here as well. Consider the characteristic system of equations

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= E_1[\rho] + E_2[\rho] \end{aligned} \tag{2.4}$$

and note that the right hand side is divergence-free for the variables (x, v). If we denote the solution of (2.4) for the initial values (x, v) on a time interval I containing 0 by $T^t(x, v)$, then the family of solution operators $\{T^t\}_{t \in I}$ (defined while $t \in I$) is known to preserve the Lebesgue measure on $\mathbb{R}^6$. Note that this family depends implicitly on the solution f; otherwise we would face a much simpler linear problem.

The equation (2.1) can be rewritten as

$$\frac{d}{dt}[f(T^t(x, v), t)] = 0 \tag{2.5}$$

and the invariance of the Lebesgue measure under $\{T^t\}$ implies that

$$\|f(\cdot, t)\|_{L^p} = \|f_0\|_{L^p} \tag{2.6}$$

for $1 \leqslant p \leqslant \infty$. In particular, we have conservation of mass and conservation of nonnegativity, just as in the classical case.

We also mention that by integrating Eq. (2.1) over the velocity space, the continuity equation

$$\rho_t + \operatorname{div}_x j = 0 \tag{2.7}$$

is satisfied, where $j(x, t) = \int vf\, dv$.

We next discuss the energy conservation law. For reasons which will become clear in subsection 2.2, we choose to also study the pure stellar dynamic Manev equation (PSM)

$$\partial_t f + v \cdot \nabla_x f + (E_2[\rho]) \cdot \nabla_v f = 0 \tag{2.8}$$

The energy conservation law for (PSM) reads

$$\frac{d}{dt}\left[\iint v^2 f(x, v, t)\, dv\, dx - \varepsilon \iint \frac{1}{|x-y|^2}\rho(y, t)\,\rho(x, t)\, dx\, dy\right] = 0 \tag{2.9}$$

For (SM), we have

$$\frac{d}{dt}\left[\iint v^2 f(x, v, t)\, dv\, dx - \iint \left(\frac{\gamma}{|x-y|} + \frac{\varepsilon}{|x-y|^2}\right)\rho(y, t)\,\rho(x, t)\, dx\, dy\right] = 0 \tag{2.10}$$

The proofs of (2.9) and (2.10) follow well known arguments. To prove (2.10), e.g., differentiate $\iint v^2 f\, dx\, dv$ with respect to t and use (2.1):

$$\frac{d}{dt}\iint v^2 f\, dx\, dv = \iint v^2(-v \cdot \nabla_x f)\, dx\, dv - \iint v^2[E_1 + E_2] \cdot \nabla_v f\, dx\, dv$$

The first term on the right is zero if f has compact support in x or vanishes sufficiently fast at infinity. After an integration by parts, the second term becomes

$$\iint 2v \cdot [E_1 + E_2] f \, dv \, dx = 2 \int [E_1 + E_2] \, j \, dx$$

$$= -2 \int \nabla [U_n + U_m] \cdot j \, dx$$

$$= 2 \int [U_n + U_m] \operatorname{div} j \, dx$$

$$= -2 \int [U_n + U_m] \, \partial_t \rho \, dx$$

$$= -\frac{d}{dt} \int [U_n + U_m] \, \rho \, dx$$

The last step in this calculation is an easy consequence of the explicit representation of U_n and U_m. Collecting terms proves (2.10).

We mention that momentum conservation also applies to (SM) and (PSM):

$$\frac{d}{dt} \iint v f(x, v, t) \, dv \, dx = 0$$

2.2. On Nonexistence of Global Solutions

We denote by

$$E_{PM}(t) = \frac{1}{2} \left[\iint v^2 f(x, v, t) \, dv \, dx - \varepsilon \iint \frac{1}{|x - y|^2} \rho(y, t) \, \rho(x, t) \, dx \, dy \right]$$

and by

$$E_M(t) = \frac{1}{2} \left[\iint v^2 f(x, v, t) \, dv \, dx - \iint \left(\frac{\gamma}{|x - y|} + \frac{\varepsilon}{|x - y|^2} \right) \times \rho(y, t) \, \rho(x, t) \, dx \, dy \right]$$

the total energies for (PSM) and (SM) respectively. In view of (2.9) and (2.10), $E_{PM}(t) = E_{PM}(0)$ for solutions of (PSM), and similarly $E_M(t) = E_M(0)$ for solutions of (SM).

Now consider a classical solution of (PSM). Following an argument first introduced by E. Horst [H], we compute the second derivative of the moment of inertia,

$$\frac{d^2}{dt^2}\int x^2\rho(x,t)\,dx = -\frac{d}{dt}\left(\iint x^2 v\cdot\nabla_x f\,dv\,dx + \iint x^2 E_2\cdot\nabla_v f\,dv\,dx\right)$$

The last integral is zero if f vanishes rapidly enough with respect to velocity. In the first integral, we integrate by parts, and use the equation again, to find

$$\begin{aligned}\frac{d^2}{dt^2}\int x^2\rho(x,t)\,dx &= -\iint 2(x\cdot v)(v\cdot\nabla_x f + E_2\cdot\nabla_v f)\,dv\,dx\\ &= \iint 2v^2 f(x,v,t)\,dv\,dx + \iint 2(x\cdot E_2)\,f(x,v,t)\,dv\,dx\end{aligned}$$

where we used integration by parts in both terms. The first term on the right is 4 times the kinetic energy. For the second term, we use the special structure of the term E_2 (see (2.3)) to compute formally

$$\begin{aligned}&-\varepsilon\int 4x\cdot\int\frac{x-y}{|x-y|^4}\rho(y,t)\,\rho(x,t)\,dy\,dx\\ &\quad= -4\varepsilon\iint\frac{1}{|x-y|^2}\rho(y,t)\,\rho(x,t)\,dx\,dy\\ &\qquad-\varepsilon\int 4y\cdot\int\frac{x-y}{|x-y|^4}\rho(y,t)\,\rho(x,t)\,dy\,dx\end{aligned}$$

and by interchanging x and y in the last term we see that

$$\iint 2(x\cdot E_2)\,f(x,v,t)\,dv\,dx = -2\varepsilon\iint\frac{1}{|x-y|^2}\rho(x,t)\,\rho(y,t)\,dx\,dy$$

which is just 4 times the potential energy! Hence we have proved that

$$\frac{d^2}{dt^2}\int x^2\rho(x,t)\,dx = 4E_{PM}(t) = 4E_{PM}(0) \tag{2.11}$$

where we have used the energy conservation law.

The identity (2.11) is most remarkable, and revealing. First, observe that the quantity $\int x^2\rho(x,t)\,dx$ is by definition nonnegative. On the other hand, if the total initial energy $E_{PM}(0)$ is negative, the time evolution of the moment of inertia is given by a downward parabola which must become negative for $t \geqslant t_0$, where t_0 can be explicitly computed in terms of the initial energy and the initial values of the moment of inertia and the quantity $\iint x \cdot vf \, dx \, dv$. It follows that the solution of (PSM) will not exist globally if $E_{PM}(0) < 0$, and the breakdown (i.e., the formation of a singularity) will happen at some time before t_0.

For the full problem (SM) as in Eq. (2.1), the same calculation yields

$$\frac{d^2}{dt^2} \int x^2 \rho(x, t) \, dx = 4E_M(0) + \iint \frac{\gamma}{|x-y|} \rho(x, t) \, \rho(y, t) \, dy \, dx \tag{2.12}$$

an identity which looks exactly as it would in the case $\varepsilon = 0$ (i.e., without the Manev correction, see [H]). Eq. (2.12) can be rewritten as

$$\frac{d^2}{dt^2} \int x^2 \rho(x, t) \, dx = 4E_{PM}(t) - \iint \frac{\gamma}{|x-y|} \rho(x, t) \, \rho(y, t) \, dy \, dx \tag{2.13}$$

It is not possible to deduce a nonexistence result directly from (2.13), as $E_{PM}(t)$ is now not invariant. However, the nonexistence result for (PSM) suggests that there will in general also *not* be a global solution for (SM): Gravitational collapse in regions of high density and low temperature is to be expected.

We conclude this subsection by mentioning that these observations are in close analogy with corresponding results for the relativistic Vlasov–Poisson system, as discussed in [GS].

2.3. Related Results for the N-Body Problem

The identity (2.11) holds, of course, in just the same way for the corresponding N-body problem

$$\dot{x}_i = v_i$$

$$\dot{v}_i = -2\varepsilon \sum_{j,\, j \neq i} \frac{x_i - x_j}{|x_i - x_j|^4}$$

and reads

$$\frac{d^2}{dt^2} \sum_i x_i^2(t) = 4E_{PM}(0) \tag{2.14}$$

In fact, (2.14) appears in [B], but its consequences are not discussed there. We note that (2.14) immediately implies that the set of phase points which lead to singularities is of positive measure: The condition $E_{PM}(0) < 0$ is a sufficient condition for this to happen.

We can use this observation to show that gravitational collapse for the N-body problem must also occur on sets of positive measure if $\gamma > 0$. Consider first the case $N = 2$ and assume that we are in a situation which leads to collapse under the pure Manev field. The following argument then shows that collapse must also happen under the full field.

Choose the origin of the coordinate system as the center of mass of the two particles, such that the sums of the particle positions and velocities, respectively, must be zero for all times before a collision. Note that the forces F_1 and F_2 in the pure Manev (F_1) and full Manev (F_2) case are acting in the radial direction, are decreasing as functions of r (where r is the distance of a particle from the origin), and satisfy $F_1(r) < F_2(r)$ for all r (the Newtonian field increases the gravitational pull).

Let $r_i(t)$, $i = 1, 2$, $t \geqslant 0$, be the solutions of

$$\frac{d^2}{dt^2} r_i = -F_i(r_i),\ r_i(0) = r_0,\ r_i'(0) = v_{r0}$$

where v_{r0} is the radial component of the initial velocity of either particle. To prove collapse, it is sufficient to show that $r_1(t) > r_2(t)$ for all $t > 0$ for which $r_2(t) > 0$.

By Taylor expansion

$$r_i(t) = r_0 + v_{r0} t - \tfrac{1}{2} F_i(r_0)\, t^2 + O(t^3)$$

and as $F_1(r_0) < F_2(r_0)$, it follows that

$$r_1(t) - r_2(t) = -\tfrac{1}{2}(F_1(r_0) - F_2(r_0))\, t^2 + O(t^3)$$

and the right-hand side will be positive for sufficiently small t. By continuity, we either have $r_1(t) > r_2(t)$ for all t for which $r_2(t)$ remains positive, or there is a minimal $T > 0$ such that $r_1(T) = r_2(T)$, and $r_1(t) > r_2(t)$ for $t < T$. But this second case leads to a contradiction because

$$\begin{aligned} r_2(T) - r_1(T) &= \int_0^T \int_0^\tau (F_1(r_1(\sigma)) - F_2(r_2(\sigma)))\, d\sigma\, d\tau \\ &\leqslant \int_0^T \int_0^\tau (F_1(r_2(\sigma)) - F_2(r_2(\sigma)))\, d\sigma\, d\tau \\ &< 0 \end{aligned}$$

where in the first estimate on the right we used the monotonicity of F_1.

In conclusion, it follows that collapse for the two-body problem with the full Manev potential must occur whenever it would occur under the pure Manev potential. As the latter happens on phase points with positive measure, so must the former.

What about N bodies? An easy argument which generalizes the idea from $N=2$ is to simply focus on two particles and assume that the remaining $N-2$ are so far away from these two that their influence remains negligible while the chosen two have time to collide in the chosen time frame. Indeed, it is not very hard to convert this idea into a rigorous argument, and it follows that the measure of the set of phase points for which gravitational collapse will occur under the full Manev potential is indeed positive. We omit the details.

3. A SYMMETRY PROPERTY

The following theorem applies to the "pure" stellar dynamic Manev case, Eq. (2.8). We discuss also some links to the blowup of solutions at singularities predicted in the previous section.

Theorem 3.1. Suppose that $f(x, v, t)$ is a solution of a Vlasov equation with intermolecular potential $U(x)=\varepsilon/|x|^2$ (e.g., PSM) on a time interval $[0, t_0]$. Then, for any $a>0$, the function

$$f_a(x, v, t) := f\left(\frac{x}{1+at}, v-a(x-vt), \frac{t}{1+at}\right) \tag{3.1}$$

is also a solution of (2.8) for $0<t<t_a$, where

$$t_a=\begin{cases}\dfrac{t_0}{1-at_0} & \text{if} \quad 0<a<1/t_0\\ \infty & \text{if} \quad a\geqslant 1/t_0\end{cases}$$

Corollary 3.2. Suppose a local solution exists for an initial value $f|_{t=0}=f_0(x, v)$, with existence interval $[0, T(f_0)]$. Then, if $a>1/T(f_0)$, there is a global solution, given by (3.1), associated with the initial value $f_0(x, v-ax)$.

Proof. By direct inspection. If we set

$$y=\frac{x}{1+at}, \quad w=v(1+at)-ax, \quad \tau=\frac{t}{1+at}$$

then

$$x = \frac{y}{1 - a\tau}, \; v = w(1 - a\tau) + ay, \; t = \frac{\tau}{1 - a\tau}$$

Let $f(x, v, t) = F(y, w, \tau)$. By the chain rule

$$(\partial_t + v\partial_x) f = \frac{1}{(1 + at)^2} (\partial_\tau + w\partial_y) F$$

Moreover,

$$\rho(x, t) = \int F(y, v(1 + at) - ax, \tau) \, dv = \frac{1}{(1 + at)^3} \int F(y, w, \tau) \, dw = \frac{\tilde{\rho}(y, \tau)}{(1 + at)^3}$$

The potential will then be

$$\begin{aligned}
\varphi(x, t) &= \int U(x - x') \, \rho(x', t) \, dx' \\
&= \frac{1}{(1 + at)^3} \int \tilde{\rho}\left(\frac{x'}{1 + at}, \tau\right) U(x - x') \, dx' \\
&= \int \tilde{\rho}(y', \tau) \; U(x - y'(1 + at)) \, dy' \\
&= \int \tilde{\rho}(y', \tau) \; U((1 + at)(y - y') \, dy'
\end{aligned}$$

By observing that $\partial w/\partial v = 1 + at$ (in the sense of a diagonal matrix), $\partial y/\partial x = 1/1 + at$, we find (with derivatives interpreted in the appropriate matrix sense)

$$\frac{\partial f}{\partial v} \frac{\partial \varphi}{\partial x} = \frac{\partial F}{\partial w} \left(\frac{\partial w}{\partial v}\right) \left[\frac{\partial \varphi}{\partial y} \frac{\partial y}{\partial x}\right] = \frac{\partial \varphi}{\partial y} \frac{\partial F}{\partial w}$$

By collecting terms, we observe that if f satisfies the general Vlasov equation with intermolecular potential U, then F satisfies the Vlasov equation

$$\partial_\tau F + w \, \partial_y F - \partial_w F \, \partial_y \varphi_a = 0$$

where $\varphi_a(y, \tau) = \int \tilde{\rho}(y', \tau) \{ (1 + at)^2 \, U[(1 + at)(y - y')] \} \, dy'$. Notice that the expression $(1 + at)$ in the last integral cancels exactly for the Manev potential. The theorem follows. ∎

We discuss the physical meaning of this theorem. To this end, we introduce the shorthand notation

$$\langle \psi(x, v) \rangle = \int \psi(x, v)\, f(x, v, t)\, dv\, dx$$

As observed in Section 2, we have the "conservation laws"

$$\frac{d}{dt} \langle x^2 \rangle = 2 \langle x \cdot v \rangle$$

$$\frac{d}{dt} \langle x \cdot v \rangle = 2E_{PM}$$

After integration,

$$\langle x^2 \rangle = \langle x^2 \rangle_0 + 2t \langle x \cdot v \rangle_0 + 2E_{PM} t^2$$

For brevity, set $E_0 = E_{PM}$ and assume that $E_0 < 0$, i.e., $E_0 = -|E_0|$. Then

$$\langle x^2 \rangle = 2E_0 \left[t^2 + 2t \frac{\langle xv \rangle_0}{2E_0} + \frac{\langle xv \rangle_0^2}{4E_0^2} \right] + \langle x^2 \rangle_0 - \frac{1}{2E_0} \langle xv \rangle_0^2$$

$$= -2\,|E_0| \left[t - \frac{\langle xv \rangle_0}{2\,|E_0|} \right]^2 + \langle x^2 \rangle_0 + \frac{1}{2\,|E_0|} \langle xv \rangle_0^2$$

Hence, the first time where $\langle x^2 \rangle$ is zero is

$$t^* = \frac{1}{2\,|E_0|} \left\{ \sqrt{\langle xv \rangle_0^2 + 2\,|E_0| \langle x^2 \rangle_0} + \langle xv \rangle_0 \right\} \tag{3.2}$$

and the time of existence of the solution will be less than t^*.

What happens to this calculation if we make the transformation given in (3.1)? Let

$$\langle xv \rangle_a = \int f_0(x, v - ax)\, xv\, dx\, dv$$

then

$$\langle xv \rangle_a = \langle xv \rangle_0 + a \langle x^2 \rangle_0$$

$$\langle v^2 \rangle_a = \langle v^2 \rangle_0 + 2a \langle xv \rangle_0 + a^2 \langle x^2 \rangle_0$$

$$\rho_a(x) = \rho_0(x)$$

and

$$E_a = \frac{1}{2}\left\{\langle v^2\rangle_a - \left\langle\!\left\langle \frac{\varepsilon}{|x-y|^2} \right\rangle\!\right\rangle\right\} = E_0 + a\langle xv\rangle_0 + \frac{1}{2}a^2\langle x^2\rangle_0$$

By repeating the calculation from above, we now find

$$\langle x^2(t)\rangle_a = \langle x^2\rangle_0 + [\langle xv\rangle_0 + a\langle x^2\rangle_0]\, 2t + \{2E_0 + 2a\langle xv\rangle_0 + a^2\langle x^2\rangle_0\}\, t^2$$

Notice that the factor in front of t^2 is twice the energy E_a. If we ask for which a this energy is zero, we find

$$a = 1/t^*$$

where t^* is the "collapse time" computed in (3.2). Hence we have the following interesting result: If $t^*(|E_0|, \langle xv\rangle_0, \langle x^2\rangle_0)$ denotes the time given by (3.2), then the function $f_0(x, v-ax)$ will have nonnegative total energy for all $a > 1/t^*$. This observation is consistent with the statement of Corollary 3.2.

4. VARIOUS PROBLEMS ASSOCIATED WITH THE PURE STELLAR DYNAMIC MANEV EQUATION

4.1. A Zero Temperature Model

Assume that the solution f of (2.8) takes the following "degenerate" form:

$$f(x, v, t) = \rho(x, t)\, \delta(v - u(x, t)) \tag{4.1}$$

(spatial density ρ, bulk velocity u, zero temperature). By taking Fourier transforms with respect to the velocity variable, one checks with little effort that (2.8) reduces to the system

$$\begin{aligned} \partial_t \rho + \sum_i \partial_{x_i}(\rho u_i) &= 0 \\ \partial_t u_i + \sum_j u_j \partial_{x_j} u_i + \partial_{x_i} U &= 0 \end{aligned} \tag{4.2}$$

where $U = U_M = -\varepsilon \int \rho(y, t)/|x-y|^2\, dy$. In fact, the transition from (2.8) to (4.2) by the ansatz (4.1) is an example for something completely general;

for a Vlasov equation with a pair potential $\Phi(|x|)$, (4.1) reduces the equation again to (4.2), but with

$$U(x, t) = \int \Phi(|x-y|)\, \rho(y, t)\, dy$$

for the Newtonian potential,

$$U = U_n = -\gamma \int \frac{\rho(y, t)}{|x-y|}\, dy \tag{4.3}$$

Remark. Global existence of classical solution of (4.2) is not to be expected even for the case $U = 0$. However, we use (4.2) to explain connections between local existence results and specific forms of the potential.

The potential U_n (and its first derivatives) are in general smoother than the input density ρ; this is a key reason why the classical Vlasov equation admits unique global classical solutions. However, the derivatives of the pure Manev potential U_M do not gain any smoothness over ρ (the Riesz transform does *not* smooth), and unpleasant consequences are to be expected. To test the situation, let us consider an even simpler, yet more singular attractive model potential

$$\Phi_m(x) = -\theta\delta(x)$$

with $\theta > 0$. This potential has smoothing properties similar to the pure Manev forces, because

$$U_m(x, t) = \int \Phi_m(x-y)\, \rho(y, t)\, dy = -\theta\rho(x, t)$$

Eq. (4.2) read

$$\begin{aligned} &\partial_t \rho + \sum_j u_j \partial_{x_j} \rho + \rho \sum_j \partial_{x_j} u_j = 0 \\ &\partial_t u_i - \theta\, \partial_{x_i} \rho + \sum_j u_j \partial_{x_j} u_j = 0 \end{aligned} \tag{4.4}$$

For our purposes it is sufficient to consider just one spatial variable. Suppose $\rho = \rho(x_1, t)$, $u_i = \delta_{i1} u(x_1, t)$. Let $x = x_1 \in \mathbb{R}$, then

$$\begin{aligned} &\rho_t + u\rho_x + \rho u_x = 0 \\ &u_t - \theta\rho_x + uu_x = 0 \end{aligned} \tag{4.5}$$

The equation determining the characteristic speeds for these equations is

$$\begin{vmatrix} u-c & \rho \\ -\theta & u-c \end{vmatrix} = (u-c)^2 + \theta\rho = 0 \tag{4.6}$$

But note that this has no real solutions for positive values of θ and ρ! Hence (4.5) is not even hyperbolic, and it follows that the Cauchy problem for the equations (4.4) is ill-posed.

This ill-posedness is also formally transparent in a linear stability analysis of (4.4) about the trivial steady solution $\rho \equiv 1$, $u \equiv 0$. To this end, take $\theta = 1$ and let $\rho = 1 + \tilde{\rho} e^{ik \cdot x + \lambda t}$, $u = \tilde{u} e^{ik \cdot x + \lambda t}$. Inserting these in (4.5) (or (4.4)) and neglecting nonlinear terms, we obtain a dispersion relation

$$\lambda^2 - k^2 = 0$$

i.e., $\lambda = \pm |k|$, for λ and k. This means that instabilities grow linearly incrementally faster as $|k| \to \infty$.

This linear stability analysis applies in the same way to the case of any integrable pair potential $\Phi(|x|)$. In this case, one finds a dispersion relation

$$\lambda^2 + k^2 \varphi(|k|) = 0 \tag{4.7}$$

with $\varphi(|k|) = \int \Phi(|x|)\, e^{ik \cdot x}\, dx$.

Remark. One can use regularized formulas for the potential,

$$U = \int [\rho(y, t) - \rho_\infty]\, \Phi(|x-y|)\, dy$$

because only the gradient of U enters in the equations.

For the Newtonian potential $\Phi_n = -\gamma/|x|$ we obtain

$$\varphi_n(|k|) = -\frac{4\pi\gamma}{|k|^2}$$

i.e., $\lambda_n(|k|) = \pm\sqrt{4\pi\gamma}$. Hence we have uniform bounds on the growth of instabilities, and the Cauchy problem for the linearized equations is well-posed. However, for the pure Manev potential $\Phi_M = -\varepsilon/|x|^2$,

$$\varphi_M(|k|) = -\frac{2\pi^2\varepsilon}{|k|}$$

hence $\lambda_M(|k|) = \pm\sqrt{2\pi^2\varepsilon}\,|k|$, and the corresponding linear Cauchy problem is ill-posed. Note that the growth of λ_M with $|k|$ is of lower order than for the more singular example (4.5).

These observations also apply to the Cauchy problem in a cube with periodic boundary conditions.

To summarize, we have observed that potentials more singular than the Newtonian potential, i.e., potentials such that $k^2\varphi(|k|) \to \infty$ as $|k| \to \infty$, seem to lead to ill-posed linearized Cauchy problems.

Do these considerations render the stellar dynamics Manev equation useless or uninteresting from a mathematical point of view? Our answer is a resounding No. The above discussion indicates that difficulties and unstable behavior of solutions are to be expected; but well-posedness for smooth data and short times remains possible, because in the above discussion:

— we considered only a very singular class of distribution functions (4.1);

— we confined ourselves to the linear stability analysis;

— and we only linearized about constant stationary solutions, whereas the case of real interest are solutions with compact support.

The point of our exercise was to show that drastic differences in terms of solvability must be expected when one includes the Manev potential. We will see below that these objections lose their strength when we assume smooth velocity distributions; and indeed, a proof of local existence and uniqueness of solutions is given in part II of our work on the stellar dynamic Manev system.

4.2. Linearization About Densities with Smooth Velocity Distribution: The Euler Equations

We now show that the situation becomes better when the velocity distribution is smooth. First, we discuss the Euler equations as a model for the Vlasov equation to demonstrate the difference between zero and non-zero temperatures. Then we study the linearized Vlasov equation about a sufficiently smooth velocity distribution function.

If we multiply the Vlasov equation by a smooth test function $\psi(v)$, we get

$$\partial_t(f, \psi) + \nabla_x \cdot (f, v\psi) + \nabla_x U \cdot (f, \nabla\psi) = 0 \tag{4.8}$$

where we have used the shorthand

$$(f, \psi) = \int f\psi \, dv$$

By setting $\psi = 1, v, v^2$, we obtain the exact (non-closed) equations for the hydrodynamic variables

$$\rho = (f, 1),\ u = \frac{1}{\rho}(f, v),\ p = \frac{1}{3}(f, v^2 - u^2)$$

A simple, non-rigorous way to close the system for ρ, u and p is to assume that f is reasonably well approximated by a Maxwellian distribution function, i.e.,

$$f \approx \rho(2\pi T)^{-3/2} \exp\left[-\frac{(v-u)^2}{2T}\right]$$

where $T = p/\rho$. We mention that this particular form of approximating f is not essential for this closure; assuming that f is approximately of the form

$$f \approx \tilde{f}(x, |v-u|, t)$$

leads to the same Euler equations, namely

$$\begin{aligned} \partial_t \rho + \sum \partial_{x_i}(\rho u_i) &= 0 \\ \partial_t u_i + \sum_j u_j \partial_{x_j} u_i + \frac{1}{\rho} \partial_{x_i} p + \partial_{x_i} U &= 0 \\ \partial_t p + \sum_i u_i \partial_{x_i} p + \frac{5}{3} p \sum_i \partial_{x_i} u_i &= 0 \end{aligned} \tag{4.9}$$

with $U = \rho * \Phi$. Note that for $p = 0$ these equations reduce to the zero-temperature case discussed above. To investigate the impact of non-zero temperature (or pressure $p = \rho T$), we now study small perturbations of the equilibrium solution $\rho_{eq} = p_{eq} = 1$, $u_{eq} = 0$. Set $\rho = 1 + \tilde{\rho}$, $p = 1 + \tilde{p}$, $u = \tilde{u}$, insert this into (4.9), neglect quadratic terms and omit the tildas. We obtain a linear system,

$$\begin{aligned} \partial_t \rho + \sum_i \partial_{x_i} u_i &= 0 \\ \partial_t u_i + \partial_{x_i}(p + U) &= 0 \\ \partial_t p + \tfrac{5}{3} \sum_i \partial_{x_i} u_i &= 0 \end{aligned} \tag{4.10}$$

Now use the ansatz $\{\rho, u, p\} = \{\rho_0, u_0, p_0\} \exp[i(k \cdot x - ct)]$ and assume for simplicity that $k = (k_1, 0, 0)$. It follows that $c = (c_1, 0, 0)$, where

$$c_1^{(0)} = 0, \; c_1^{(+)} = \sqrt{\tfrac{5}{3} + \varphi(|k|)}, \; c_1^{(-)} = -\sqrt{\tfrac{5}{3} + \varphi(|k|)}$$

Hence, for the Manev potential, the "increment of instability" reads

$$\begin{aligned}\Re\lambda(k) &= \left[\Im \sqrt{\frac{5}{3} + \varphi(|k|)} \right] |k| \\ &= |k| \sqrt{\frac{2\pi^2\varepsilon}{|k|} - \frac{5}{3}} = \sqrt{|k| \, (2\pi^2\varepsilon - \frac{5}{3} |k|)}\end{aligned}$$

whenever $|k| \leqslant 6\pi^2\varepsilon/5$. It follows that there is a critical wave number $k_{crit} = 6\pi^2\varepsilon/5$, such that all modes with $k > k_{crit}$ are stable. This shows that for nonzero temperatures the instability does not necessarily lead to ill-posedness of the Cauchy problem.

4.3. Linearization About Smooth Velocity Distributions: The Full Equation

Consider, once again, a general Vlasov equation, and set

$$f(x, v, t) = f_0(v) + \tilde{f}(x, v, t)$$

We assume that $f_0 \in C_1(\mathbb{R}^3)$ and that $\tilde{f}$ is a small perturbation. After using this ansatz in the Vlasov equation, we neglect quadratic terms and drop the tildas. The result is

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f_0 = 0$$

where $U = \rho * \Phi$ and $\rho = \int f \, dv$. The ansatz

$$f(x, v, t) = f_0(v) \exp(ik \cdot x + \lambda t)$$

leads to a dispersion relation for $\lambda = \lambda(k)$:

$$1 = \varphi(|k|) \int \frac{(ik \cdot \nabla_v) \, f_0(v) \, dv}{\lambda + ik \cdot v}$$

For $u \in \mathbb{R}$, let $F(u) = \int f_0(v) \, \delta(u - k \cdot v/|k|) \, dv$. Then

$$1 = \varphi(|k|) \int \frac{i \, |k| \, F'(u) \, du}{\lambda + i \, |k| \, u}$$

or, with $c(|k|) = i\lambda/|k|$,

$$1 = \varphi(|k|) \int \frac{F'(u)\, du}{u - c(|k|)}$$

Integrating by parts, we get

$$1 = \varphi(|k|) \int \frac{F(u)\, du}{(u-c)^2} \tag{4.11}$$

Suppose now that $c = c_0 + ic_1$ with $c_1 > 0$ (i.e., we consider an unstable mode). Eq. (4.11) decomposes as follows by splitting into real and imaginary parts:

$$\begin{aligned} 1 &= \varphi(|k|) \int (u^2 - c_1^2) \frac{F(c_0 + u)}{[u^2 + c_1^2]^2}\, du \\ 0 &= \int \frac{uF(c_0 + u)}{[u^2 + c_1^2]^2}\, du \end{aligned} \tag{4.12}$$

We estimate the "increment of instability" $\Re\lambda(|k|) = |k|\, c_1(|k|)$ as $|k| \to \infty$. The first equation from (4.12) implies that

$$1 \leqslant \frac{|\varphi(k)|}{c_1} \|F\|_\infty \int \frac{|u^2 - 1|}{(u^2+1)^2}\, du = \frac{2\, |\varphi(k)|\, \|F\|_\infty}{c_1}$$

and we find

$$|\Re\lambda(|k|)| \leqslant 2\, \|F\|_\infty\, |k|\, |\varphi(|k|)|$$

i.e., the "increment of instability" will be bounded if $\sup_k |k|\, |\varphi(|k|)| < \infty$. We have implicitly proved the following.

Proposition 4.1. Suppose that $f_0 \in C^1(\mathbb{R}^3)$ has compact support, and let $d(f_0) = \operatorname{diam}(\operatorname{supp} f_0)$. Then all eigenvalues of the linearized pure stellar dynamic Manev equation

$$\lambda(k)\, \tilde{f}_k + ik \cdot v\tilde{f}_k + ik \cdot \nabla_v f_0 \left[\tilde{\rho}_k \frac{2\pi^2 \varepsilon}{|k|} \right] = 0$$

where $\tilde{f}_k = \tilde{f}_k(v)$ and $\tilde{\rho}_k = \int \tilde{f}_k(v)\, dv$, satisfy the inequality

$$|\Re\lambda(k)| \leqslant [2\pi d(f_0)]^2\, \varepsilon\, \|f_0\|_\infty$$

5. A REMARKABLE SCALING PROPERTY

In this last section we discuss the relevance of Boltzmann collision terms for the general interaction potential

$$U(r) = \alpha/r^n, n > 1 \tag{5.1}$$

(for the discussion in this section, we assume that the forces are repulsive; attractive forces will be discussed in part three of our series of papers. As will there be shown, the addition of a Boltzmann collision term alone does not quite do justice to the attractive potential—there are also reasons to add a Smoluchowski-type coagulation term). We have so far discussed the "pure stellar dynamic Manev equation" with a Vlasov type interaction term associated with $n = 2$. Let us now include a Boltzmann collision term in our considerations and investigate how the various terms in such an equation behave under rescalings.

If m denotes the molecular mass, the differential cross section associated with (5.1) can be written as

$$\sigma(u, \theta) = \left(\frac{\alpha}{mu^2}\right)^{2/n} g_n(\cos\theta)$$

where g_n is a function such that $\int_{-1}^{1} g_n(\mu)(1-\mu)\, d\mu < \infty$. (see Cercignani [C]). The Boltzmann collision kernel is then $B(u, \theta) = |u|\, \sigma(u, \theta)$, and the Boltzmann collision integral is defined for all $n > 1$. For $1 < n < 3$, we define the general Vlasov force term for the potential (5.1) by

$$\vec{F} = -\nabla_x \int \frac{\alpha}{|x-y|^n} \rho(y, t)\, dy \underset{\text{def}}{=} -\int \frac{\alpha}{|y|^n} \nabla\rho(x-y, t)\, dy \tag{5.2}$$

For the range $1 < n < 3$, and from a formal point of view, we can therefore consider the Vlasov-Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f + \frac{\vec{F}}{m} \cdot \nabla_v f = Q(f, f) \tag{5.3}$$

with $Q(f, f) = \iint (\alpha/m\, |v - v_*|^2)^{2/n} |\, v - v_* \,|\, g_n(\cos\theta)\{f'f'_* - ff_*\}\, dn\, dv_*$. We investigate the question how the Boltzmann and Vlasov terms compare in significance. To this end, let x_0, v_0 and t_0 be typical length, velocity and time scales related by $x_0 = v_0 t_0$. Suppose furthermore that $v_0^2 = T_0/m$, and that ρ_0 is a typical value of the spatial density. We pass to a dimensionless form of (5.3) by setting $\tilde{x} = x/x_0$, $\tilde{v} = v/v_0$, $\tilde{t} = t/t_0$, and

$$f(x, v, t) = \rho_0 v_0^{-3} \tilde{f}(\tilde{x}, \tilde{v}, \tilde{t}) \tag{5.4}$$

We insert (5.4) into (5.3), use the chain rule and collect terms. After deleting the tildas, we find

$$\partial_t f + v \cdot \nabla_x f - C_V(x_0)\, \nabla_x \int \frac{\rho(y,t)}{|x-y|^n}\, dy \cdot \nabla_v f = C_B(x_0)\, Q_1(f,f) \quad (5.5)$$

with $Q_1(f,f) = \iint |v - v_*|^{1-4/n}\, g_n(\cos\theta)\{f'f'_* - ff_*\}\, dn\, dv_*$,

$$C_V(x_0) = \rho_0 \left(\frac{\alpha}{mv_0^2}\right) x_0^{3-n}$$

and

$$C_B(x_0) = \rho_0 \left(\frac{\alpha}{mv_0^2}\right)^{2/n} x_0$$

Let $\lambda = C_V(x_0)/C_B(x_0)$, i.e., $\lambda = (\alpha/mv_0^2)^{(n-2)/n}\, x_0^{2-n}$. If we set $r_* = [\alpha/mv_0^2]^{1/n}$, then $\lambda = [r_*/x_0]^{n-2}$. While x_0 is a typical length for the density distribution function in question, r_* will be a distance between two particles which are strongly correlated. If $\rho_0^{-1/3}$ is interpreted as a typical distance between two particles, then a kinetic model for the particle system is only meaningful if $x_0 \gg \rho_0^{-1/3}$ and $r_* \ll \rho_0^{-1/3}$. Hence, if we set $\varepsilon := r_*/x_0$, we have $\varepsilon \ll 1$, and

$$\lambda = \frac{C_V(x_0)}{C_B(x_0)} = \varepsilon^{n-2} \sim \begin{cases} 0 & \text{if} \quad n > 2 \\ 1 & \text{if} \quad n = 2 \\ \infty & \text{if} \quad n < 2. \end{cases}$$

We see that the case $n = 2$ is the only situation where the Vlasov and Boltzmann terms will be of the same order of magnitude. For the ordinary Coulomb potential, the Vlasov term is dominant, and for $n > 2$ the Boltzmann term is dominant.

We plan to address the attractive case in a future publication. In this case, coagulation integral operators are meaningful in addition to the Vlasov and Boltzmann collision terms.

APPENDIX: A JUSTIFICATION OF THE MANEV CORRECTION

Consider the motion of a particle in a central force field as described by the Lagrangian formalism. The Lagrangian is

$$L = \frac{mv^2}{2} - U(r) \quad \text{(A.1)}$$

where $r = |x|$ and $\dot{x} = v$. We only discuss the case where $U(r) \leqslant 0$, easily enforced in the case of attractive Newtonian forces. The equations of motion are

$$\frac{d}{dt}\frac{\partial}{\partial v_i} L = \frac{\partial}{\partial x_i} L$$

In polar coordinates $x = (r, \varphi)$,

$$v^2 = \dot{r}^2 + r^2 \dot{\varphi}^2 \tag{A.2}$$

we realize that the angular momentum $P_\varphi = \partial L/\partial \varphi_t = m r^2 \varphi_t = M = \text{const.}$ is invariant because φ is a cyclic coordinate, i.e., $\partial L/\partial \varphi = 0$. We use this conservation of angular momentum to eliminate φ from the equation of motion for r.

The easiest way to do this is by using the conservation of energy. The total energy is

$$T = v \cdot \nabla_v L - L = \frac{mv^2}{2} + U(r)$$

hence

$$\frac{m}{2}\dot{r}^2 + \frac{m}{2} r^2 \dot{\varphi}^2 = T - U(r)$$

and with $\dot{\varphi}^2 = M^2/m^2 r^4$,

$$\frac{m}{2}\dot{r}^2 = T - U(r) - \frac{M^2}{2mr^2} \tag{A.3}$$

Note that if $U(r) = -\gamma/r$ with $\gamma > 0$, Eq. (A.3) implies the classical result that r cannot become zero unless $M = 0$. Indeed, the left hand side of (A.3) remains nonnegative for all times, hence the right hand side must also remain nonnegative, hence we can deduce a lower bound (the perihelion) for r. We find it convenient to set

$$U_{\text{eff}}(r) = U(r) + \frac{M^2}{2mr^2}$$

and (A.3) becomes

$$\frac{m}{2}\dot{r}^2 = T - U_{\rm eff}(r) \tag{A.4}$$

We shall refer to (A.4) as the "radial equation."

Our goal now is to understand what changes occur in (A.4) if relativistic effects are taken into account. The Lagrangian for a free particle in classical mechanics reads

$$L_{\rm cl}^{(0)} = \frac{mv^2}{2}$$

whereas its relativistic counterpart is

$$L_{\rm rel}^{(0)} = -mc^2\sqrt{1 - \frac{v^2}{c^2}}$$

If we substitute this in the Lagrangian (A.1), we obtain

$$L = -mc^2\sqrt{1 - v^2/c^2} - U(r) \tag{A.5}$$

and after switching to polar coordinates again, the previous discussion can be repeated with this new Lagrangian. The angular momentum and its invariance now read

$$P_\varphi = \frac{\partial L}{\partial \varphi_t} = mr^2\varphi_t/\sqrt{1 - v^2/c^2} = M = \text{const}$$

and energy and its invariance are

$$E = mc^2 + T = \frac{mc^2}{\sqrt{1 - v^2/c^2}} + U(r) = \text{const}$$

From the last equation we obtain

$$\sqrt{1 - (v/c)^2} = \frac{mc^2}{E - U(r)}$$

hence $\varphi_t = (Mc^2/r^2)(1/(E - U(r)))$, and we can expand

$$\begin{aligned}1-(v/c)^2 &= 1-\frac{\dot{r}^2}{c^2}-\frac{r^2\dot{\varphi}^2}{c^2}\\ &= 1-\frac{\dot{r}^2}{c^2}-\frac{M^2c^2}{r^2}\frac{1}{(E-U(r))^2}\\ &= \frac{m^2c^4}{(E-U(r))^2}\end{aligned}$$

Solving the last equation for $\dot{r}$, we find

$$\dot{r}^2 = c^2\left\{1-\frac{1}{(E-U)^2}\left[\frac{M^2c^2}{r^2}+(mc^2)^2\right]\right\}$$

Let $E = T + mc^2$, then this new radial equation can be rewritten in the form

$$\frac{1}{2}m\dot{r}^2 = \frac{1}{(1+(T-U)/mc^2)^2}\left\{T-U-\frac{M^2}{2mr^2}+\frac{(T-U)^2}{2mc^2}\right\} \tag{A.6}$$

Note that (A.6) reduces to (A.3) for $c = \infty$. Let $\varepsilon = T/mc^2$ and assume that $|\varepsilon| \ll 1$. Eq. (A.6) becomes

$$\frac{1}{2}m\dot{r}^2 = \frac{1}{(1+\varepsilon-U/mc^2)^2}\left\{T\left(1+\frac{\varepsilon}{2}\right)-U(1+\varepsilon)-\frac{M^2}{2mr^2}+\frac{U^2}{2mc^2}\right\}$$

Little qualitative effects are to be expected from the ε-perturbations, as ε is small. However, independently of the size of ε, the last term $U^2/2mc^2$ can alter the dynamics significantly when r is such that $U(r) \approx mc^2$. To anticipate such effects, we set $\varepsilon = 0$ and arrive at the simplified equation

$$\frac{1}{2}m\dot{r}^2 \approx \frac{1}{(1-U/mc^2)^2}\left\{T-U\left[1-\frac{U}{2mc^2}\right]-\frac{M^2}{2mr^2}\right\} \tag{A.7}$$

We discuss the part of the denominator on the right for a non-positive potential $U = -|U| \leqslant 0$. To this end, we rewrite (A.7) as

$$\frac{1}{2}m\dot{r}^2 \approx \frac{mc^2}{2}\left\{\left(\frac{|U|/(mc^2)}{1+|U|/(mc^2)}\right)^2+2\,\frac{T+|U|-M^2/(2mr^2)}{mc^2(1+|U|/(mc^2))^2}\right\}$$

Suppose that $|U| = \theta mc^2$, where θ is fixed, and let (formally) $c \to \infty$. The right hand side of the last identity will in this limit be bounded by

$$\frac{mc^2}{2}\left[\left(\frac{\theta}{1+\theta}\right)^2 + 2\frac{\theta}{(1+\theta)^2}\right] = \frac{mc^2}{2}\left(1 - \frac{1}{(1+\theta)^2}\right)$$

and we realize that the main part of the denominator in (A.7) is to guarantee that $|\dot{r}| < c$ even for $|U| \to \infty$. In a nonrelativistic approximation where such a speed constraint is not required, we can replace this denominator by 1.

Finally, we focus on a Newtonian potential $U(r) = -\alpha/r$, with $\alpha > 0$, and discuss qualitative effects arising from the corrections in the right-hand side of (A.7). The numerator there then reads

$$A(r) := T + \frac{\alpha}{r} + \frac{1}{2mr^2}\left[\left(\frac{\alpha}{c}\right)^2 - M^2\right]$$

Let $r_{\min}$ be defined by $A(r_{\min}) = 0$. For $c = \infty$, $r_{\min}$ is always well defined and coincides with the perihelion distance defined earlier in the nonrelativistic case. However, $r_{\min}$ does not exist if $c < \infty$ and

$$|M| < M_{\text{critical}} := \alpha/c$$

A particle with $0 \leqslant |M| < M_{\text{critical}}$ will reach the origin $x = 0$ after a finite time.

In conclusion, the behavior of this "quasirelativistic" model with a potential U is qualitatively similar to a corresponding classical model with the modified potential

$$U_{\text{mod}}(r) = U(r) - \frac{U^2(r)}{2mc^2}$$

In the Newtonian case,

$$U_{\text{mod}}(r) = -\frac{\alpha}{r} - \frac{\alpha^2}{2mc^2}\frac{1}{r^2}$$

this differs from the Manev potential for a central force field,

$$U_{\text{Manev}}(r) = -\frac{\alpha}{r} - \frac{3}{2}\frac{\alpha^2}{mc^2}\frac{1}{r^2}$$

only by a numerical factor. As we are largely interested in qualitative effects, we do not discuss possible explanations of this numerical factor.

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